

A GRIFFITH CRACK MODEL IN A GENERALIZED NONHOMOGENEOUS INTERLAYER OF BONDED DISSIMILAR HALF-PLANES

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The Griffith crack problem in bonded dissimilar half-planes is examined. To eliminate the unrealistic oscillatory stress near the interface crack tips, the interfacial transition zone is modeled by a very thin nonhomogeneous interlayer whose elastic properties vary continuously between the bonded materials and adhesive material. The interlayer thickness is assumed to be the sum of the maximum heights of asperities at the two bonded material surfaces. The crack problem is reduced to a set of Cauchy integral equations which can be solved numerically. The applicability of the generalized nonhomogeneous interlayer model is investigated by comparing it with the classical interface crack model.

Keywords: generalized nonhomogeneous interlayer, Griffith crack, stress intensity factors, fracture mechanics

1. Introduction

Interfaces are intrinsic to many modern composite materials since they are always layered. The structural performance of such materials is generally dependent on their interfaces which are heterogeneities such as discontinuities in elastic and thermal properties as well as residual stresses. Fracture mechanics of layered materials has been extensively used to characterize the initiation and propagation of delamination (Hutchinson and Suo, 1992; Suo, 1990; Wang *et al.*, 2021; Zhang and Wang, 2016). The classical model for an idealized (or perfect) bonding bimaterial structure containing an interface crack was established by Williams (1959) by assuming the zero-thickness interface, which means that the stress and displacement vary continuously across the interface. However, analytical solutions for interface crack problems show that there is an oscillatory singularity which is physically unreasonable and results in material interpenetration near the ends of the interface crack (England, 1965; Erdogan, 1965; Williams, 1959). In order to eliminate the unrealistic oscillatory singularity, a closed crack tip model was developed by Comninou (1977) based on classical solutions which assumed that the surfaces of interface crack contact was frictionless near the tips. This model was further applied to interfacial fracture analysis of anisotropic materials (Ayatollahi *et al.*, 2022; Herrmann and Loboda, 1999), piezoelectric materials (Govorukha *et al.*, 2000; Sheveleva *et al.*, 2015), thermopiezoelectric materials (Qin and Mai, 1999). A modified interface dislocation model for interface fracture analysis was presented by Zhang and Wang (2013), which represented an inverse square-root singularity at the interface crack tips and avoided the oscillatory behavior.

The idealized interface model for the fracture problem in bonded dissimilar homogeneous materials may be too unrealistic from the micromechanical point of view, and it does not capture any effect of mechanical characteristics of the real transition layer (i.e., interlayer) between the materials on the stress and displacement distribution. In fact, an interlayer forms whatever the actual mechanism of binding is, and should be taken into account as a distinct chemical species or a distinct phase. Physical properties in a thin interlayer are highly nonhomogeneous with the steeply varying composition profile. The layer thickness ranges from nanometers to fractions of a millimeter (Yang and Shih, 1994). Two attractive features emerge when the nonhomogeneous interlayer model is used to the interface crack problems. Firstly, the stress oscillatory singularity is removed so that the local mode mixity is independent of the distance ahead the crack tip. Secondly, the conventional stress intensity factors (SIFs) can be defined as the crack problems in a homogeneous medium. The nonhomogeneous interlayer model considering the interpenetration or interdiffusion of molecules in the interfacial zone was first theoretically developed by Delale and Erdogan (1988) and successfully applied to fracture problems in bimaterial structures (Erdogan *et al.*, 1991; Ozturk and Erdogan, 1995). A more generalized interlayer model introducing a distribution parameter independent of interlayer thickness and material properties was presented by Wang *et al.* (1996, 1997), and the Erdogan's interlayer model could be obtained when the distribution parameter tends to infinity.

On the other hand, there exist various roughnesses and asperities at each bonded material surface, and the third material used as an adhesive may be filled with gaps between the surfaces of two primary components as shown in Fig. 1a. As a result, a more generalized nonhomogeneous interlayer containing adhesive materials may also be emerged instead of the Erdogan's interlayer. The elastic constants of this nonhomogeneous interlayer are not only dependent on the physical properties of bonded materials but also on those of the adhesive material. Thus, the purpose of this paper is to develop a theoretical model of the Griffith crack in a generalized nonhomogeneous interlayer considering the influence of surface roughness of bonded materials and elastic properties of the adhesive material.

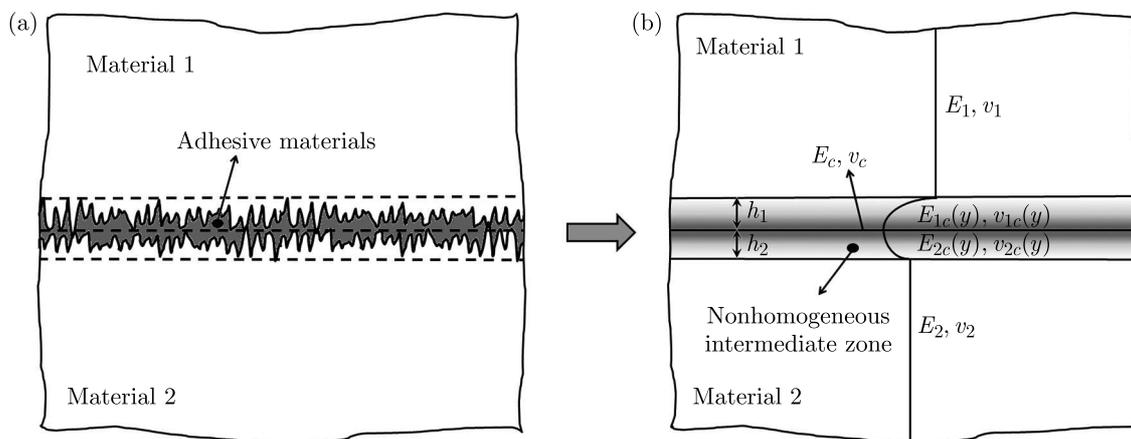


Fig. 1. The generalized nonhomogeneous interlayer model: (a) microstructures of the bonding zone between two homogeneous elastic half-planes, (b) the effective nonhomogeneous interlayer

2. Mathematical model for a Griffith crack in a generalized nonhomogeneous interlayer

2.1. Formulation of the generalized nonhomogeneous interlayer

Consider an interface mechanical problem in a bimaterial structure composed of two isotropic and homogeneous materials with elastic moduli E_1 and E_2 , and Poisson's ratios ν_1 and ν_2 . The

thickness of the nonhomogeneous interlayer is assumed to be the sum of the maximum heights of asperities at the two bonded material surfaces, assuming that the adhesive layer is much thinner than the interlayer. It is noted that the assumption of the zero-thickness adhesive layer is consistent with the classical model of interface mechanics (England, 1965; Erdogan, 1965; Williams, 1959). In order to facilitate the complicated interfacial transition zone, a generalized nonhomogeneous interlayer model with the thickness of $h = h_1 + h_2$ is developed as shown in Fig. 1b. The mechanical properties of the interlayer may vary steeply, but it is crucial to maintain continuity at the interfaces with the adjacent materials. Delale and Erdogan (1988) introduced an interlayer model with material constants that exhibited an exponential variation, which ensured an inverse square-root singularity at the crack tips and made the problem analytically tractable. Therefore, in this paper, the material parameters along such an interlayer are assumed to be only dependent on y , and the elastic modulus has a form of

$$E_{jc}(y) = E_c e^{\lambda_j y} \tag{2.1}$$

where $j = 1, 2$ and E_c is the elastic modulus of the adhesive material, subscripts 1c and 2c are corresponding to regions $0 \leq y \leq h_1$ and $-h_2 \leq y < 0$, respectively. The parameter λ_j can be determined from the continuity conditions for $E_{1c}(h_1) = E_1$ and $E_{2c}(-h_2) = E_2$, that is

$$\lambda_j = \begin{cases} \frac{1}{h_1} \ln \frac{E_1}{E_c} & 0 \leq y \leq h_1 \\ -\frac{1}{h_2} \ln \frac{E_2}{E_c} & -h_2 \leq y < 0 \end{cases} \tag{2.2}$$

Introducing Airy stress functions $F_{jc}(x, y)$ as

$$\sigma_{xx}^{jc} = \frac{\partial^2 F_{jc}(x, y)}{\partial y^2} \quad \sigma_{yy}^{jc} = \frac{\partial^2 F_{jc}(x, y)}{\partial x^2} \quad \sigma_{xy}^{jc} = -\frac{\partial^2 F_{jc}(x, y)}{\partial x \partial y} \tag{2.3}$$

Considering that the functions E_{jc} and ν_{jc} are independent of x , the compatibility condition is expressed as

$$\begin{aligned} \nabla^4 F_{jc}(x, y) - 2\lambda_j \frac{\partial}{\partial y} \nabla^2 F_{jc}(x, y) + \lambda_j^2 \frac{\partial^2 F_{jc}(x, y)}{\partial y^2} \\ - \left[\frac{d^2 \nu_{jc}(y)}{dy^2} - 2\lambda_j \frac{d\nu_{jc}(y)}{dy} + \lambda_j^2 \nu_{jc}(y) \right] \frac{\partial^2 F_{jc}(x, y)}{\partial x^2} = 0 \end{aligned} \tag{2.4}$$

It is known that Poisson's ratios do not influence the SIFs significantly (Delale and Erdogan, 1983; 1988). Thus, we further assume that

$$\begin{aligned} \nabla^4 F_{jc}(x, y) - 2\lambda_j \frac{\partial}{\partial y} \nabla^2 F_{jc}(x, y) + \lambda_j^2 \frac{\partial^2 F_{jc}(x, y)}{\partial y^2} = 0 \\ \frac{d^2 \nu_{jc}(y)}{dy^2} - 2\lambda_j \frac{d\nu_{jc}(y)}{dy} + \lambda_j^2 \nu_{jc}(y) = 0 \end{aligned} \tag{2.5}$$

By means of Eq. (2.5)₂, Poisson's ratios are obtained as follows

$$\nu_{jc}(y) = (\nu_c + \nu_{j0} y) e^{\lambda_j y} \tag{2.6}$$

where $\nu_{10} = (E_c \nu_1 - E_1 \nu_c) / (E_1 h_1)$ and $\nu_{20} = -(E_c \nu_2 - E_2 \nu_c) / (E_2 h_2)$. Now, the elastic properties and geometric dimensions of the generalized nonhomogeneous interlayer are totally determined.

2.2. Griffith crack model in the generalized nonhomogeneous interlayer

Attention is now focused on the Griffith crack problem with a generalized nonhomogeneous interlayer as shown in Fig. 2, where the physical length of the crack is designated by $2c$. The origin of the rectangular coordinate system x - y is fixed at the middle point of the crack and the x -axis coincides with the crack line. The elastic properties are constant for $y > h_1$ and $y < -h_2$, and the plane elasticity problem can be formulated by assuming that each material is perfectly bonded along the planes $y = h_1$, $y = 0$ and $y = -h_2$ except for the crack.

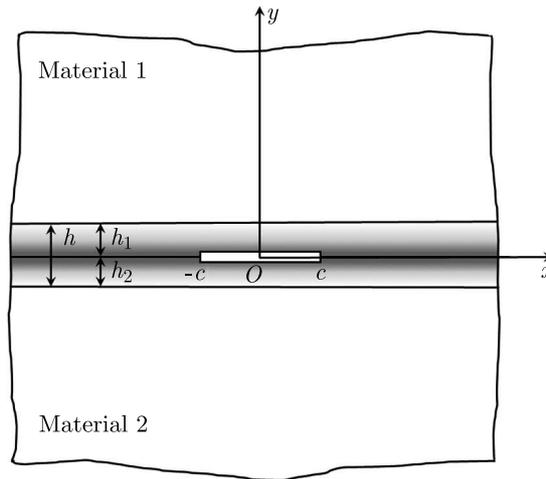


Fig. 2. Geometry of the Griffith crack problem with a generalized nonhomogeneous interlayer

By using the Fourier transform technique, the solutions for Airy stress functions $F_{jc}(x, y)$ in the generalized nonhomogeneous interlayer and $F_j(x, y)$ in the homogeneous materials have forms of

$$F_{jc}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ [A_{j1}(\xi) + A_{j2}(\xi)y]e^{m_{j1}y} + [A_{j3}(\xi) + A_{j4}(\xi)y]e^{m_{j2}y} \} e^{-i\xi x} d\xi \quad (2.7)$$

$$F_j(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [B_{j1}(\xi) + B_{j2}(\xi)y] e^{-\delta_j|\xi||y-i\xi x} d\xi$$

where $A_{jk}(\xi)$ ($k = 1, 2, 3, 4$) and $B_{jl}(\xi)$ ($l = 1, 2$) are unknown functions, m_{jk} and δ_j are defined as

$$m_{j1} = m_{j3} = \frac{\lambda_j}{2} - \sqrt{\xi^2 + \frac{\lambda_j^2}{4}} \quad m_{j2} = m_{j4} = \frac{\lambda_j}{2} + \sqrt{\xi^2 + \frac{\lambda_j^2}{4}} \quad (2.8)$$

$$\delta_j = \begin{cases} -1 & y > h_1 \\ 1 & y < -h_2 \end{cases}$$

By substituting Airy stress functions (2.7) into Eq. (2.3) and using the constitutive equations and strain-displacement relations, the stress fields and displacements in the bimaterial structure are obtained as follows

$$\begin{aligned} \sigma_{yy}^{jc}(x, y) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^2 [(A_{j1} + A_{j2}y)e^{m_{j1}y} + (A_{j3} + A_{j4}y)e^{m_{j2}y}] e^{-i\xi x} d\xi \\ \sigma_{xy}^{jc}(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi \{ [m_{j1}A_{j1} + (1 + m_{j1}y)A_{j2}] e^{m_{j1}y} \\ &\quad + [m_{j2}A_{j3} + (1 + m_{j2}y)A_{j4}] e^{m_{j2}y} \} e^{-i\xi x} d\xi \end{aligned} \tag{2.9}$$

$$\begin{aligned} \sigma_{xx}^{jc}(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ [m_{j1}^2 A_{j1} + (2 + m_{j1}y)m_{j1}A_{j2}] e^{m_{j1}y} \\ &\quad + [m_{j2}^2 A_{j3} + (2 + m_{j2}y)m_{j2}A_{j4}] e^{m_{j2}y} \} e^{-i\xi x} d\xi \\ \sigma_{yy}^j(x, y) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^2 (B_{j1} + B_{j2}y) e^{-\delta_j |\xi| y - i\xi x} d\xi \\ \sigma_{xy}^j(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi [-\delta_j |\xi| B_{j1} + (1 - \delta_j |\xi| y) B_{j2}] e^{-\delta_j |\xi| y - i\xi x} d\xi \\ \sigma_{xx}^j(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [\xi^2 B_{j1} + (\xi^2 y - 2\delta_j |\xi|) B_{j2}] e^{-\delta_j |\xi| y - i\xi x} d\xi \end{aligned} \tag{2.10}$$

$$\begin{aligned} u_x^{jc}(x, y) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\xi E_{jc}(y)} \{ [(m_{j1}^2 + \xi^2 \nu_{jc}(y)) A_{j1} \\ &\quad + [(2 + m_{j1}y)m_{j1} + \xi^2 y \nu_{jc}(y)] A_{j2}] e^{m_{j1}y} + [(m_{j2}^2 + \xi^2 \nu_{jc}(y)) A_{j3} \\ &\quad + [(2 + m_{j2}y)m_{j2} + \xi^2 y \nu_{jc}(y)] A_{j4}] e^{m_{j2}y} \} e^{-i\xi x} d\xi \\ u_y^{jc}(x, y) &= -\frac{1}{2\pi E_c} \int_{-\infty}^{\infty} \left\{ \left[\left(\frac{\xi^2}{m_{j1} - \lambda_j} e^{-\lambda_j y} + m_{j1}(\nu_c + \nu_{j0}y) - \nu_{j0} \right) A_{j1} \right. \right. \\ &\quad + \left. \left(\frac{(m_{j1} - \lambda_j)y - 1}{(m_{j1} - \lambda_j)^2} \xi^2 e^{-\lambda_j y} + (\nu_c + \nu_{j0}y)m_{j1}y + \nu_c \right) A_{j2} \right] e^{m_{j1}y} \\ &\quad + \left[\left(\frac{\xi^2}{m_{j2} - \lambda_j} e^{-\lambda_j y} + m_{j2}(\nu_c + \nu_{j0}y) - \nu_{j0} \right) A_{j3} \right. \\ &\quad + \left. \left. \left(\frac{(m_{j2} - \lambda_j)y - 1}{(m_{j2} - \lambda_j)^2} \xi^2 e^{-\lambda_j y} + (\nu_c + \nu_{j0}y)m_{j2}y + \nu_c \right) A_{j4} \right] e^{m_{j2}y} \right\} e^{-i\xi x} d\xi \end{aligned} \tag{2.11}$$

$$\begin{aligned} u_x^j(x, y) &= -\frac{1}{2\pi E_j} \int_{-\infty}^{\infty} \frac{1}{i\xi} \left\{ (1 + \nu_j) \xi^2 B_{j1} + [(1 + \nu_j) \xi^2 y - 2\delta_j |\xi| y] B_{j2} \right\} e^{-\delta_j |\xi| y - i\xi x} d\xi \\ u_y^j(x, y) &= \frac{1}{2\pi E_j} \int_{-\infty}^{\infty} \left\{ (1 + \nu_j) \delta_j |\xi| B_{j1} + [(1 + \nu_j) \delta_j |\xi| y - \nu_j + 1] B_{j2} \right\} e^{-\delta_j |\xi| y - i\xi x} d\xi \end{aligned} \tag{2.12}$$

The boundary conditions of the Griffith crack problem are expressed as

$$\begin{aligned} u_x^{1c}(x, 0) &= u_x^{2c}(x, 0) & u_y^{1c}(x, 0) &= u_y^{2c}(x, 0) & |x| &> c \\ \sigma_{yy}^{1c}(x, 0) &= \sigma_{yy}^{2c}(x, 0) & \sigma_{xy}^{1c}(x, 0) &= \sigma_{xy}^{2c}(x, 0) & |x| &< \infty \\ \sigma_{yy}^{1c}(x, 0) &= -p(x) & \sigma_{xy}^{1c}(x, 0) &= -q(x) & |x| &\leq c \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} u_x^{1c}(x, h_1) &= u_x^1(x, h_1) & u_y^{1c}(x, h_1) &= u_y^1(x, h_1) & |x| < \infty \\ \sigma_{yy}^{1c}(x, h_1) &= \sigma_{yy}^1(x, h_1) & \sigma_{xy}^{1c}(x, h_1) &= \sigma_{xy}^1(x, h_1) & |x| < \infty \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} u_x^{2c}(x, -h_2) &= u_x^2(x, -h_2) & u_y^{2c}(x, -h_2) &= u_y^2(x, -h_2) & |x| < \infty \\ \sigma_{yy}^{2c}(x, -h_2) &= \sigma_{yy}^2(x, -h_2) & \sigma_{xy}^{2c}(x, -h_2) &= \sigma_{xy}^2(x, -h_2) & |x| < \infty \end{aligned} \quad (2.15)$$

where $p(x)$ and $q(x)$ are known functions.

2.3. Integral equations and stress intensity factors

We now introduce functions $f_j(x)$ at the crack plane

$$f_1(x) = \frac{\partial}{\partial x}[u_x^{1c}(x, 0) - u_x^{2c}(x, 0)] \quad f_2(x) = \frac{\partial}{\partial x}[u_y^{1c}(x, 0) - u_y^{2c}(x, 0)] \quad (2.16)$$

The stress and displacement on the crack line for a bimaterial structure can be obtained based on Eqs. (2.9) and (2.11), then the functions $A_{jk}(\xi)$ are determined by making use of boundary conditions (2.13)-(2.15), and after some lengthy manipulations, we have

$$A_{jk}(\xi) = \gamma_{k1}^{jc}(\xi)g_1(\xi) + \gamma_{k2}^{jc}(\xi)g_2(\xi) \quad (2.17)$$

where $j = 1, 2$, $k = 1, 2, 3, 4$, and

$$g_1(\xi) = E_c \int_{-\infty}^{\infty} f_1(t)e^{i\xi t} dt \quad g_2(\xi) = E_c \int_{-\infty}^{\infty} f_2(t)e^{i\xi t} dt \quad (2.18)$$

The functions $\gamma_{k1}^{jc}(\xi)$ and $\gamma_{k2}^{jc}(\xi)$ are not given here due to tediousness, and can be obtained by solving 12 linear algebraic equations in terms of $g_1(\xi)$ and $g_2(\xi)$ based on the boundary conditions. By substituting Airy functions Eqs. (2.7), and Eq. (2.17) into Eq. (2.3), the stresses on the crack line are obtained as follows

$$\begin{aligned} \sigma_{yy}^{1c}(x, 0) &= -\frac{E_c}{2\pi} \int_{-c}^c \int_{-\infty}^{\infty} [P_{11}(\xi)f_1(t) + P_{12}(\xi)f_2(t)]e^{i\xi(t-x)} d\xi dt \\ \sigma_{xy}^{1c}(x, 0) &= -\frac{E_c}{2\pi} \int_{-c}^c \int_{-\infty}^{\infty} [P_{21}(\xi)f_1(t) + P_{22}(\xi)f_2(t)]e^{i\xi(t-x)} d\xi dt \end{aligned} \quad (2.19)$$

where

$$\begin{aligned} P_{11}(\xi) &= \xi^2[\gamma_{11}^{1c}(\xi) + \gamma_{31}^{1c}(\xi)] & P_{12}(\xi) &= \xi^2[\gamma_{12}^{1c}(\xi) + \gamma_{32}^{1c}(\xi)] \\ P_{21}(\xi) &= -i\xi[m_{11}\gamma_{11}^{1c}(\xi) + \gamma_{21}^{1c}(\xi) + m_{12}\gamma_{31}^{1c}(\xi) + \gamma_{41}^{1c}(\xi)] \\ P_{22}(\xi) &= -i\xi[m_{11}\gamma_{12}^{1c}(\xi) + \gamma_{22}^{1c}(\xi) + m_{12}\gamma_{32}^{1c}(\xi) + \gamma_{42}^{1c}(\xi)] \end{aligned} \quad (2.20)$$

It should be noted that $P_{11}(\xi)$, $P_{22}(\xi)$ are even functions and $P_{12}(\xi)$, $P_{21}(\xi)$ are odd functions with respect to ξ . In addition, the following asymptotic properties of $P_{11}(\xi)$, $P_{12}(\xi)$, $P_{21}(\xi)$ and $P_{22}(\xi)$ are further given

$$\lim_{\xi \rightarrow +\infty} P_{12}(\xi) = \lim_{\xi \rightarrow +\infty} P_{21}(\xi) = -\frac{1}{4} \quad \lim_{\xi \rightarrow +\infty} P_{11}(\xi) = \lim_{\xi \rightarrow +\infty} P_{22}(\xi) = 0 \quad (2.21)$$

Therefore, the Griffith crack problem with a generalized nonhomogeneous interlayer can be reduced to the first kind singular integral equations based on the analysis of Eq. (2.21) rather than the second kind ones which represent physically unreasonable stress oscillatory singularity and lead to overlapping near the ends of the crack surfaces. Using boundary condition Eq. (2.13)₃, the Cauchy singular integral equations are obtained as follows

$$\begin{aligned} \frac{E_c}{\pi} \int_{-c}^c [Q_{11}(t, x) f_1(t) + Q_{12}(t, x) f_2(t)] dt - \frac{E_c}{4\pi} \int_{-c}^c \frac{f_2(t)}{t-x} dt &= p(x) \\ \frac{E_c}{\pi} \int_{-c}^c [Q_{21}(t, x) f_1(t) + Q_{22}(t, x) f_2(t)] dt - \frac{E_c}{4\pi} \int_{-c}^c \frac{f_1(t)}{t-x} dt &= q(x) \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} Q_{11}(t, x) &= \int_0^\infty P_{11}(\xi) \cos[\xi(t-x)] d\xi & Q_{12}(t, x) &= \int_0^\infty \left[iP_{12}(\xi) + \frac{1}{4} \right] \sin[\xi(t-x)] d\xi \\ Q_{21}(t, x) &= \int_0^\infty \left[iP_{21}(\xi) + \frac{1}{4} \right] \sin[\xi(t-x)] d\xi & Q_{22}(t, x) &= \int_0^\infty P_{22}(\xi) \cos[\xi(t-x)] d\xi \end{aligned} \tag{2.23}$$

It is clear from condition (2.13)₁ that

$$\int_{-c}^c f_1(t) dt = 0 \qquad \int_{-c}^c f_2(t) dt = 0 \tag{2.24}$$

Integral Eqs. (2.22)-(2.24) can be solved numerically by using the method developed by Erdogan (1975). Furthermore, the undetermined functions $f_1(x)$ and $f_2(x)$ are of conventional inverse square-root singularity at $x = \pm c$ according to the singular integral equation theory. Normalizing the interval $(-c, c)$ by changing variables as $t = c\tilde{t}$ and $x = c\tilde{x}$, integral equations of (2.22)-(2.24) have solutions in the following form

$$f_1(t) = \frac{H_1(\tilde{t})}{\sqrt{1-\tilde{t}^2}} = \frac{\sum_{i=0}^{n-1} D_{1i} T_i(\tilde{t})}{\sqrt{1-\tilde{t}^2}} \qquad f_2(t) = \frac{H_2(\tilde{t})}{\sqrt{1-\tilde{t}^2}} = \frac{\sum_{i=0}^{n-1} D_{2i} T_i(\tilde{t})}{\sqrt{1-\tilde{t}^2}} \tag{2.25}$$

where $H_1(\tilde{t})$ and $H_2(\tilde{t})$ are continuous bounded functions defined in the interval $|\tilde{t}| \leq 1$, $T_i(\tilde{t})$ is the first kind Chebyshev polynomial, and the coefficients D_{1i} and D_{2i} are constants as yet to be determined. After discretization, the singular integral equations can be rewritten as

$$\begin{aligned} \frac{E_c}{n} \sum_{k=1}^n \left[Q_{11}(\tilde{t}_k, \tilde{x}_r) H_1(\tilde{t}_k) + Q_{12}(\tilde{t}_k, \tilde{x}_r) H_2(\tilde{t}_k) - \frac{H_2(\tilde{t}_k)}{4(\tilde{t}_k - \tilde{x}_r)} \right] &= p(\tilde{x}_r) \\ \frac{E_c}{n} \sum_{k=1}^n \left[Q_{21}(\tilde{t}_k, \tilde{x}_r) H_1(\tilde{t}_k) + Q_{22}(\tilde{t}_k, \tilde{x}_r) H_2(\tilde{t}_k) - \frac{H_1(\tilde{t}_k)}{4(\tilde{t}_k - \tilde{x}_r)} \right] &= q(\tilde{x}_r) \\ \sum_{k=1}^n H_1(\tilde{t}_k) = 0 & \qquad \sum_{k=1}^n H_2(\tilde{t}_k) = 0 \end{aligned} \tag{2.26}$$

where the discretization points \tilde{t}_k and \tilde{x}_r are defined by

$$\begin{aligned} \tilde{t}_k &= \cos\left(\frac{2k-1}{2n}\pi\right) & k &= 1, 2, \dots, n \\ \tilde{x}_r &= \cos\left(\frac{r}{n}\pi\right) & r &= 1, 2, \dots, n-1 \end{aligned} \tag{2.27}$$

The equation system in (2.26) includes $2n$ linear algebraic equations in terms of $H_1(\tilde{t}_k)$ and $H_2(\tilde{t}_k)$, and the coefficients D_{1i} and D_{2i} can be easily solved. The SIFs at the crack tip are of interest and are defined as

$$\begin{aligned} K_1(c) &= \sqrt{2\pi(x-c)}\sigma_{yy}^{1c}(x,0) = -\frac{E_c}{4}\sqrt{\pi c}\sum_{i=1}^{n-1} D_{2i} \\ K_2(c) &= \sqrt{2\pi(x-c)}\sigma_{xy}^{1c}(x,0) = -\frac{E_c}{4}\sqrt{\pi c}\sum_{i=1}^{n-1} D_{1i} \end{aligned} \quad (2.28)$$

where the following integral property of the Chebyshev polynomial is used

$$\frac{1}{\pi} \int_{-1}^1 \frac{1}{\tilde{t} - \tilde{x}} \frac{T_i(\tilde{t})}{\sqrt{1-\tilde{t}^2}} dt = -\frac{|\tilde{x}|}{\tilde{x}\sqrt{\tilde{x}^2-1}} \left(\tilde{x} - \frac{|\tilde{x}|}{\tilde{x}} \sqrt{\tilde{x}^2-1} \right)^i \quad |\tilde{x}| > 1 \quad (2.29)$$

The definition of fracture mode mixity is

$$\psi = \arctan \frac{K_2(c)}{K_1(c)} \quad (2.30)$$

The energy release rate for crack propagation at the crack tip can then be calculated by using the crack closure concept as

$$G(c) = \frac{1}{E_c} [K_1^2(c) + K_2^2(c)] \quad (2.31)$$

3. Numerical results and discussions

The influence of elastic property and thickness of the generalized nonhomogeneous interlayer on the SIFs is investigated in numerical examples. The material combination used is as follows (Delale and Erdogan, 1988)

$$\begin{aligned} E_1 &= 20.685 \cdot 10^{10} \frac{\text{N}}{\text{m}^2} & \nu_1 &= 0.3 \\ E_2 &= 6.895 \cdot 10^{10} \frac{\text{N}}{\text{m}^2} & \nu_1 &= 0.3 \end{aligned} \quad (3.1)$$

Poisson's ratio of the adhesive material is $\nu_c = 0.3$ since it has very little effect on the SIFs. Without loss of any generality, we set

$$p(x) = p_0 \quad q(x) = q_0 \quad (3.2)$$

in the following numerical analysis. It is noted that the developed model for Griffith crack problems in bonded dissimilar elastic half-planes can be reduced to the interfacial region model (Delale and Erdogan, 1988) and the classical interface crack model (Williams, 1959) if the elastic modulus of the adhesive material is given as $E_c = E_1(E_2/E_1)^{h_1/h}$ and the thickness $h \rightarrow 0$, respectively. To verify validity of the presented theoretical model, the normalized SIFs for a Griffith crack with the thickness ratio $h_1/h = 0.5$ are given in Table 1, and the corresponding results calculated by Delale and Erdogan (1988) are also listed. The results show that the present scheme achieves a good agreement of the accuracy.

Both Figs. 3 and 4 plot the influence of the elastic modulus of the adhesive material and thickness ratio h_1/c on the normalized SIFs at the crack tip with $K_0 = \pi c \sqrt{p_0^2 + q_0^2}$, for different combinations of $p(x)$ and $q(x)$. It is found that pure far-field uniform tension can produce mode II

Table 1. Normalized SIFs calculated based on the generalized nonhomogeneous interlayer model and the interfacial region model for $h_1/h = 0.5$

c/h	Delale and Erdogan (1988)		Present results	
	$K_1(c)/(p_0\sqrt{\pi c})$	$K_2(c)/(p_0\sqrt{\pi c})$	$K_1(c)/(p_0\sqrt{\pi c})$	$K_2(c)/(p_0\sqrt{\pi c})$
0.1	1.004	0.023	1.0041	0.0257
0.25	1.014	0.058	1.0141	0.0584
0.5	1.026	0.092	1.0263	0.0924
1	1.036	0.127	1.0360	0.1223
5	1.046	0.169	1.0452	0.1661

SIF and a pure far-field uniform shear loading can also produce mode I SIF at the crack tip based on the generalized nonhomogeneous interlayer model. This behavior is similar to the classical interface crack problem. In addition, it may be observed that the normalized SIFs tend to increase as the elastic modulus of the adhesive material increases, and the thickness ratio h_1/c has a very significant effect on the crack-tip stress fields especially for small values of h_1/c .

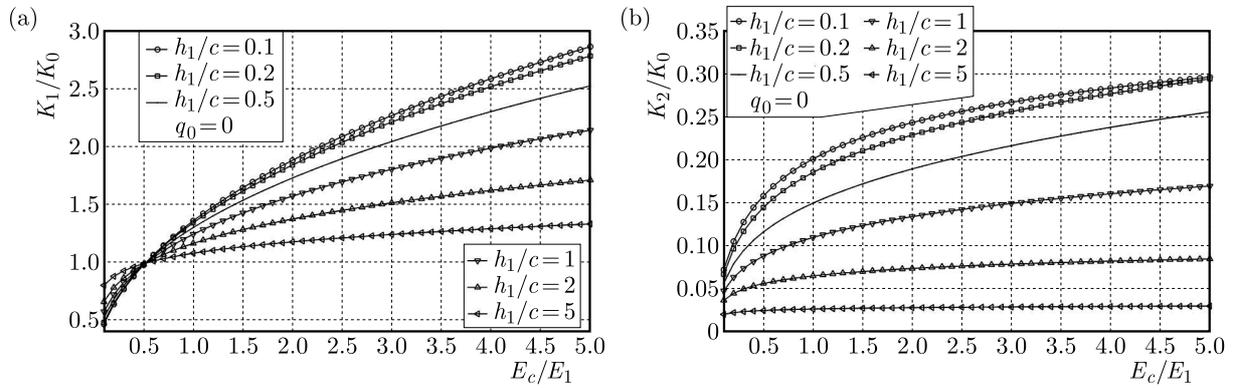


Fig. 3. Effect of the elastic modulus of the adhesive material on normalized (a) mode I and (b) mode II SIFs, with $p(x) = p_0$, and $q(x) = 0$

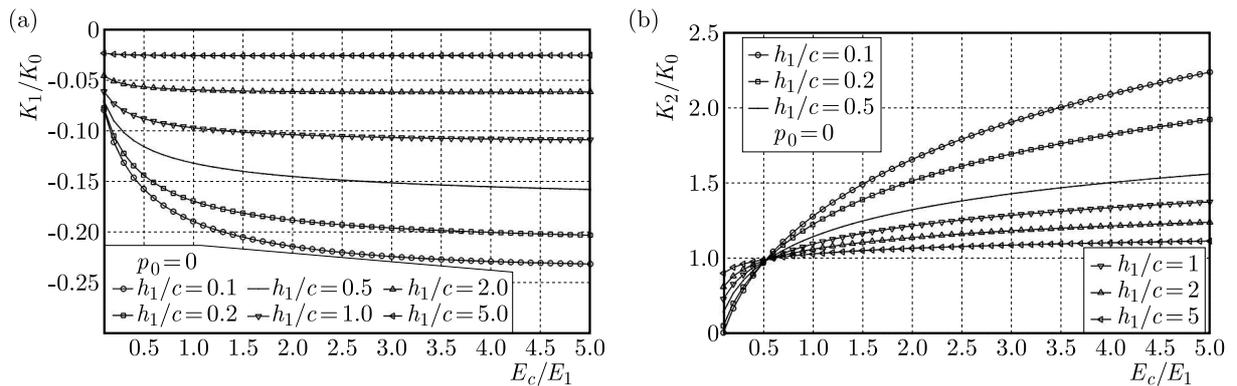


Fig. 4. Effect of the elastic modulus of the adhesive material on normalized (a) mode I and (b) mode II SIFs, with $p(x) = 0$, and $q(x) = q_0$

It is worth noting that the developed theoretical model introduces three new parameters, namely h_1 , h_2 and E_c , which have clear physical significance and can be measured. The singularity obtained based on the classical interface crack model may result in a complex number (as shown in Eq. (3.3)) and lead to stress oscillations and displacement interference at the crack tips. However, the energy release rate is largely uninfluenced and has great practical guidance for

the criteria of crack propagation. To better understand the numerical results discussed below, it may be worthwhile to recall the crack-tip fields in the classical interface fracture mechanics. The SIFs at crack tip are given as (Sun and Jih, 1987)

$$K_1(c) + iK_2(c) = (p_0 + iq_0)\sqrt{\pi c}(1 + 2i\varepsilon)(2c)^{-i\varepsilon} \quad (3.3)$$

where

$$\begin{aligned} \varepsilon &= \frac{1}{2\pi} \ln \frac{1 - \beta}{1 + \beta} & \beta &= \frac{\mu_2(\kappa_1 - 1) - \mu_1(\kappa_2 - 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)} \\ \kappa_j &= 3 - 4\nu_j & \mu_j &= \frac{E_j}{2(1 + \nu_j)} \quad j = 1, 2 \end{aligned} \quad (3.4)$$

The energy release rate for crack propagation in the interface is (Malyshev and Salganik, 1965)

$$G(c) = \frac{1 - \beta^2}{E^*} [K_1^2(c) + K_2^2(c)] \quad (3.5)$$

with $1/E^* = (1/E_1 + 1/E_2)/2$, and $\bar{E}_j = E_j/(1 - \nu_j^2)$ for plane strain, $\bar{E}_j = E_j$ for plane stress.

The SIFs obtained by Eqs. (2.28) cannot be directly compared to those calculated using the classical interface crack approach due to different singularities at the crack tips. However, the energy release rates calculated based on the developed model can also offer an important reference, similar to the classical interface crack model. The elastic modulus E_c of the adhesive should be predetermined in real adhesively bonded components when the developed model in this paper is used. On the other hand, if E_c is regarded as a more generalized parameter that depends only on the physical properties of bonded materials, the applicability of the model may be broader. Furthermore, the fracture criteria based on the energy release rates in the classical interface crack problems agree well with the experimental results. Therefore, the elastic modulus of the adhesive material can be determined as

$$E_c = \frac{E^*}{1 - \beta^2} \quad (3.6)$$

by comparing with Eqs. (2.31) and (3.5). Figure 5 shows the influence of h_1/c and E_c on the normalized energy release rates G/G_0 , where $G_0 = 10^{-10}(p_0^2 + q_0^2)c$. It can be seen that values of G/G_0 obtained based on both models have little difference for the pure tension case when Eq. (3.6) holds, especially for small values of h_1/c .

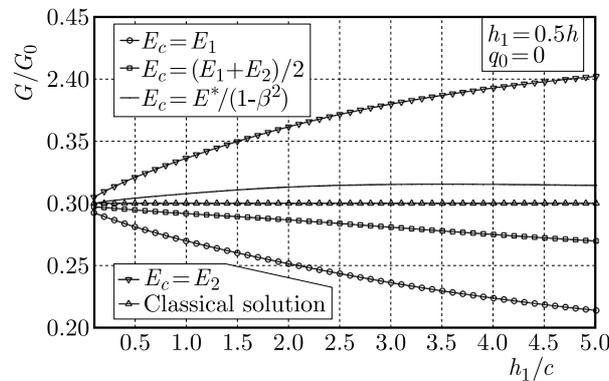


Fig. 5. Effect of the length ratio h_1/c between the generalized nonhomogeneous interlayer and crack on the normalized energy release rate

The influence of loading ratio q_0/p_0 for different thickness ratios of the interlayer h_1/h on the normalized SIFs, mode mixity and normalized energy release rate are plotted in Fig. 6 with

$E_c = E^*/(1-\beta^2)$. The results indicate that the mode II SIF K_2/K_0 and mode mixity ψ increase, while mode I SIF K_1/K_0 decreases with an increasing value of q_0/p_0 . These tendencies are similar to the experimental results of Liechti and Chai (1992). From Fig. 6d, we can see that differences in the energy release rate G/G_0 between the generalized nonhomogeneous interlayer model and classical interface crack model are minimal when the thickness ratio of the interlayer is 0.75. Moreover, these differences can be further significantly reduced by selecting an appropriate value of h_1/h between 0.5 and 0.75.

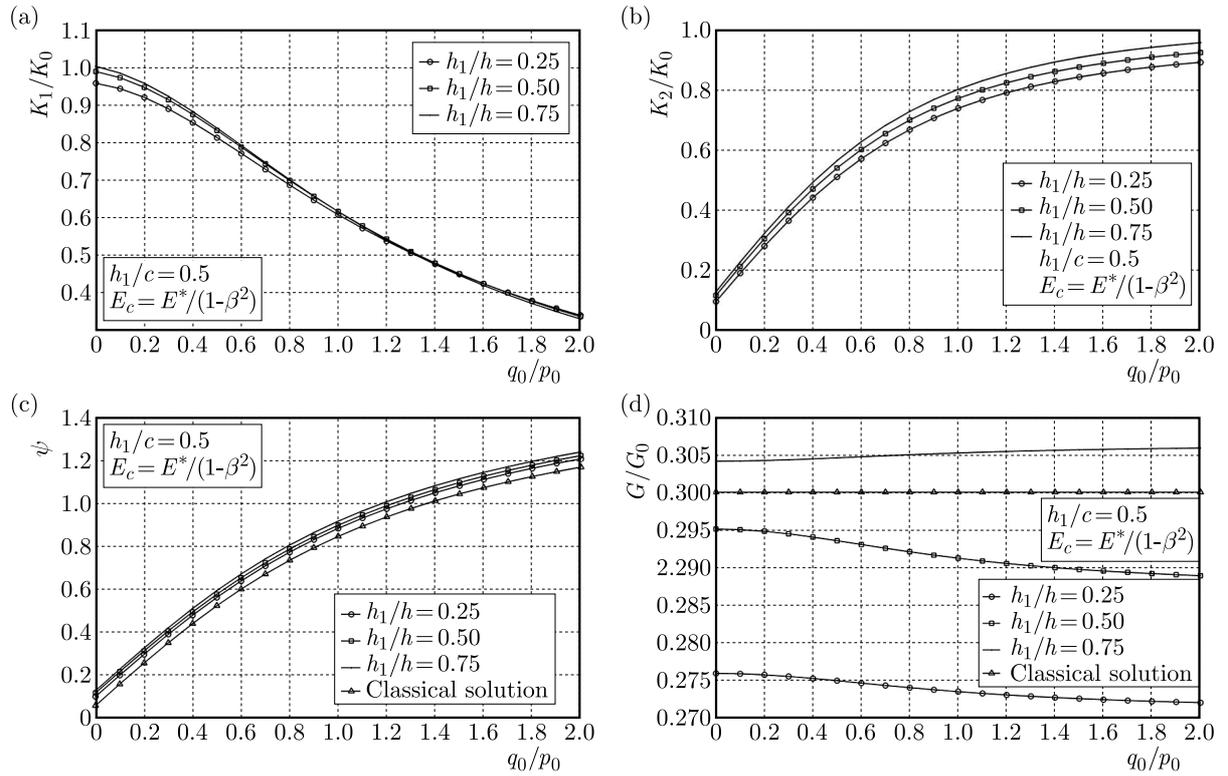


Fig. 6. Effect of the loading ratio q_0/p_0 on (a) normalized mode I SIF, (b) normalized mode II SIF, (c) mode mixity, and (d) normalized energy release rate

Finally, it should be noted that dimensions of SIFs of the classical interface crack model given by Eq. (3.3) depend on a complex factor $i\epsilon$, hence, it is challenging to employ SIFs to develop a suitable fracture criterion for interface crack problems. However, the SIFs determined using Eqs. (2.28) overcome this limitation and can be used to establish the SIF-based fracture criterion. A quasi mode I (or quasi mode II) crack can be defined when the crack is subjected to simple tension (or pure shear) at a remote distance since the singular crack tip field is dominated by mode I (or mode II) fracture. Therefore, the SIF-based fracture criteria for quasi mode I and II cracks respectively take the form

$$K_1 = K_{1C} \quad K_2 = K_{2C} \quad (3.7)$$

where K_{1C} and K_{2C} are the critical SIFs to be determined by experiments. For the mixed mode crack, the criterion may be taken in the elliptical form as

$$\left(\frac{K_1}{K_{1C}}\right)^2 + \left(\frac{K_2}{K_{2C}}\right)^2 = 1 \quad (3.8)$$

4. Conclusions

This paper proposes a Griffith crack model for two bonded dissimilar homogeneous isotropic elastic half-planes, taking into account roughness at each material surface and the effect of the adhesive material. A generalized nonhomogeneous interlayer is developed to model the adhesive interface, where it is assumed that all material properties vary continuously between those of the bonded materials and the adhesive material, and depend only on exponential functions of the coordinate y (perpendicular to the interface). The Griffith crack problem is then reduced to a set of singular integral equations which can be solved numerically. The influence of elastic property and thickness of the interlayer on mode I and II SIFs, mode mixity and energy release rate is studied through numerical results. The applicability of the developed crack model with the generalized nonhomogeneous interlayer is also investigated by comparing it with the classical interface crack model. It is found that the energy release rates calculated by the two models are very close when the elastic modulus and geometric dimensions of the generalized nonhomogeneous interlayer are appropriately selected.

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